

LIPSCHITZ SLICES AND THE DAUGAVET EQUATION FOR LIPSCHITZ OPERATORS

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ABSTRACT. We introduce a substitute for the concept of slice for the case of non-linear Lipschitz functionals and transfer to the non-linear case some results about the Daugavet and the alternative Daugavet equations previously known only for linear operators.

1. INTRODUCTION

Our paper is motivated by a recent paper [16] by X. Huang, D. Tan, and R. Wang, where the study of numerical range is extended to non-linear Lipschitz operators. They show, among other interesting results that we shall comment on, that the connection between the Daugavet and the alternative Daugavet equations and the numerical range also holds for Lipschitz maps.

The aim of this note is to show that introducing a reasonable substitute for the concept of slice for the case of non-linear Lipschitz functionals and using ideas from [1], one can transfer to the non-linear case several results about the Daugavet and the alternative Daugavet equations previously known only for linear operators.

Let us settle the notation and present the definitions and known results which will be relevant to our discussion. Given a Banach space X over \mathbb{K} ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), we write S_X for its unit sphere and B_X for its closed unit ball. The dual space of X is denoted by X^* and $L(X)$ is the Banach algebra of all bounded linear operators from X to X . The space X has the *Daugavet property* [7] if every rank-one operator $T \in L(X)$ satisfies the *Daugavet equation*

$$\|\text{Id} + T\| = 1 + \|T\|. \quad (\text{DE})$$

In this case, all operators on X which do not fix copies of ℓ_1 (in particular, weakly compact operators) also satisfy (DE) [14]. If every rank-one operator $T \in L(X)$ satisfies the *alternative Daugavet equation*

$$\max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| = 1 + \|T\| \quad (\text{aDE})$$

(\mathbb{T} being the set of scalars of modulus one), X has the *alternative Daugavet property* [12] and then all operators on X which do not fix copies of ℓ_1 (in particular, weakly compact operators) also satisfy (aDE) [1]. Let us remark that, by a convexity argument, T satisfies (DE) (resp. (aDE)) if and only if λT satisfies (DE) (resp. (aDE)) for every $\lambda > 0$. A Banach space has *numerical index 1* [5] if every $T \in L(X)$ satisfies that $v(T) = \|T\|$, where

$$v(T) = \{|x^*(Tx)| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\}$$

Date: July 14th, 2014.

2010 Mathematics Subject Classification. Primary 46B04. Secondary 46B80, 46B22, 47A12.

Key words and phrases. Numerical radius, numerical index, Daugavet equation, Daugavet property, SCD space, Lipschitz operator.

The work of the first named author was partially done during his visit to the University of Granada in June and July 2013 under the support of Spanish MICINN and FEDER project no. MTM2012-31755. The second and third authors have been partially supported by Spanish MICINN and FEDER project no. MTM2012-31755 and by Junta de Andalucía and FEDER grants FQM-185 and P09-FQM-4911.

is the *numerical radius* of the operator T . It is known [5] that

$$v(T) = \|T\| \iff T \text{ satisfies (aDE)}.$$

Thus, X has numerical index 1 if and only if every $T \in L(X)$ satisfies (aDE).

The space of all Lipschitz functions $T: X \rightarrow Y$ will be equipped with the seminorm

$$\|T\| = \sup \left\{ \frac{\|T(x_1) - T(x_2)\|}{\|x_1 - x_2\|} : x_1 \neq x_2 \in X \right\}. \quad (1)$$

Remark that for a linear operator this seminorm coincides with the standard operator norm. If one wants to make this space a Banach space one may quotient out the kernel of the seminorm (1), i.e., the subspace of constant functions, or equivalently, one may consider the Banach space $\text{Lip}_0(X, Y)$ consisting of all Lipschitz functions on X that vanish at 0, with the Lipschitz constant (1) as the actual norm. These two procedures indeed lead to isometric versions of the Banach space of Lipschitz functions which we denote by $\text{Lip}(X, Y)$; by $\text{Lip}(X)$ we denote $\text{Lip}(X, X)$. The question of which procedure to choose is just a matter of taste; maybe the approach using $\text{Lip}_0(X, Y)$ is a little easier to visualise.

For a map $T \in \text{Lip}(X, Y)$ we denote

$$\text{slope}(T) := \left\{ \frac{T(x_1) - T(x_2)}{\|x_1 - x_2\|} : x_1 \neq x_2 \in X \right\};$$

Observe that if $T \in L(X)$, then $\text{slope}(T) = T(S_X)$.

In their paper [16] (see also [15]), X. Huang, D. Tan, and R. Wang defined a new concept of numerical range for Lipschitz maps (which extends the classical one in the case when the map is a linear operator), and the corresponding notion of Lipschitz numerical index of a Banach space, as follows. The *numerical range* of $T \in \text{Lip}(X)$ is defined as

$$V(T) := \left\{ \frac{f(Tx - Ty)}{\|x - y\|^2} : x, y \in X, x \neq y, f \in X^*, \text{Re } f(x - y) = \|f\| \|x - y\| = \|x - y\|^2 \right\}$$

and the *numerical radius* of T is $v(T) = \sup\{|\lambda| : \lambda \in V(T)\}$. Observe that in the case when T is a linear operator, then the numerical range (and so the numerical radius) coincides with the classical definition. They demonstrated for a Lipschitz map T on a Banach space X the following statements, previously known for linear operators:

$$\|\text{Id} + T\| = 1 + \|T\| \iff \sup \text{Re } V(T) = \|T\|$$

and

$$\max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| = 1 + \|T\| \iff v(T) = \|T\|.$$

The *Lipschitz numerical index* of X is also defined in [16] as

$$n_L(X) = \inf \{v(T) : T \in \text{Lip}(X), \|T\| = 1\}.$$

Observe that we always have that $n_L(X) \leq n(X)$, and it is proved in [16] that equality holds for spaces with the Radon-Nikodým property and also that $n_L(X) = 1$ for a real lush space X . (The notion of a lush space will be recalled in Section 4.) Besides, they also investigated stability properties of the Lipschitz numerical index, extending to this new setting known results of the (linear) numerical index for c_0 -, ℓ_1 - and ℓ_∞ -sums and for some vector valued function spaces.

As we have already mentioned, our goal in this paper is to transfer some known results for linear operators on spaces with the Daugavet property or the alternative Daugavet property to Lipschitz operators. To do so, we introduce the Lipschitz version of the concept of a slice of the unit sphere and prove that this new concept shares some fundamental properties of linear slices (see the fundamental Lemma 2.4) which allow to show that well-known geometrical characterizations of the Daugavet and the alternative

Daugavet properties in terms of slices are also true for Lipschitz slices. This is the content of Section 2. With these results as tools, we are able to extend in Section 3 some ideas of the paper [1] to Lipschitz operators, showing that in a Banach space with the Daugavet property (respectively with the alternative Daugavet property), those Lipschitz operators whose slopes are Radon-Nikodým sets, or Asplund sets, or CPCP sets, or do not contain copies of ℓ_1 , satisfy (DE) (resp. (aDE)). As a consequence of this, we show that a Banach space X with numerical index 1 has Lipschitz numerical index 1 provided it is Asplund, or it has the Radon-Nikodým property, or it has the CPCP, or it does not contain copies of ℓ_1 (for the Radon-Nikodým property, this result appeared in [16]). Finally, Section 4 is devoted to show that (complex) lush spaces have Lipschitz numerical index 1, extending [16, Theorem 2.6] which was only proved for the real case.

Let us finish this introduction presenting some common notation. For a set A of a Banach space X , $\text{conv}(A)$ and $\overline{\text{conv}}(A)$ stand for the convex hull and the closed convex hull of A , respectively, and $\text{conv}(\mathbb{T}A)$ and $\overline{\text{conv}}(\mathbb{T}A)$ are the absolutely convex hull and the absolutely closed convex hull of A , respectively. By $\text{Re}(\cdot)$ we denote the real part function, understanding that it is just the identity if we are dealing only with real numbers.

2. SLICES AND LIPSCHITZ SLICES

Let X be a Banach space. A *slice* of a set $A \subset X$ is a non-empty intersection of A with an open half-space. In other words a slice of A is a non-empty set of the form

$$\{x \in A: \text{Re } x^*(x) > \alpha\}, \quad (2)$$

where x^* is a non-zero continuous linear functional and $\alpha \in \mathbb{R}$. Slices of the unit sphere play a crucial role in the geometric approach to the Daugavet and alternative Daugavet equations, thanks to the following two results.

Lemma 2.1 (a particular case of [7, Lemma 2.2]). *A Banach space X has the Daugavet property if and only if for every $x \in S_X$, every $\varepsilon > 0$ and every slice S of S_X , there is $y \in S$ such that $\|x + y\| > 2 - \varepsilon$.*

Lemma 2.2 ([12, Proposition 2.1]). *A Banach space X has the alternative Daugavet property if and only if for every $x \in S_X$, every $\varepsilon > 0$ and every slice S of S_X , there is $y \in S$ such that $\max_{\theta \in \mathbb{T}} \|x + \theta y\| > 2 - \varepsilon$.*

We introduce a notion of slice generated by a Lipschitz functional which will play a fundamental role in our discussion; note that non-linear slices were also considered in [13].

Definition 2.3. Let X be a Banach space. A *Lip-slice* of S_X is a non-empty set of the form

$$\left\{ \frac{x_1 - x_2}{\|x_1 - x_2\|} : x_1 \neq x_2, \frac{f(x_1) - f(x_2)}{\|x_1 - x_2\|} > \alpha \right\},$$

where $f \in \text{Lip}(X, \mathbb{R})$ is non-zero and $\alpha \in \mathbb{R}$. The following notation will be useful: for $f \in \text{Lip}(X, \mathbb{R}) \setminus \{0\}$ and $\varepsilon > 0$, we write

$$S(S_X, f, \varepsilon) := \left\{ \frac{x_1 - x_2}{\|x_1 - x_2\|} : x_1 \neq x_2, \frac{f(x_1) - f(x_2)}{\|x_1 - x_2\|} > \|f\| - \varepsilon \right\}$$

and observe that this is never empty and so it is a Lip-slice of S_X ; conversely, every Lip-slice of S_X can be written in this form.

Remark that for a real-linear functional $f = \text{Re } x^*$ with $x^* \in X^*$, the above definition gives a usual slice of S_X

$$S(S_X, \text{Re } x^*, \varepsilon) := \{x \in S_X: \text{Re } x^*(x) > \|x^*\| - \varepsilon\},$$

which agrees with the formula (2).

The next result shows that slices generated by Lipschitz functionals behave similarly to those generated by linear functionals.

Lemma 2.4 (Fundamental lemma). *Let X be a Banach space, $f \in \text{Lip}(X, \mathbb{R})$, $\varepsilon > 0$, and $A \subset S_X$. If $\overline{\text{conv}}(A) \cap S(S_X, f, \varepsilon) \neq \emptyset$ then $A \cap S(S_X, f, \varepsilon) \neq \emptyset$.*

We need a preliminary result which shows that every rescaling of a functional has the same Lipschitz slices.

Lemma 2.5. *Let $f \in \text{Lip}(X, \mathbb{R})$ and $\varepsilon > 0$. Then, for every $r > 0$ the functional defined by $f_r(x) = \frac{1}{r}f(rx)$ for $x \in X$ satisfies $\|f_r\| = \|f\|$ and*

$$S(S_X, f_r, \varepsilon) = S(S_X, f, \varepsilon).$$

Proof. Fix $r > 0$ and $x, y \in X$ with $x \neq y$, and observe that

$$\frac{f_r(x) - f_r(y)}{\|x - y\|} = \frac{f(rx) - f(ry)}{\|rx - ry\|}.$$

Using this it is immediate that $\|f_r\| = \|f\|$. Besides, if $\frac{x-y}{\|x-y\|} \in S(S_X, f_r, \varepsilon)$ then, by the above equality, we have that

$$\frac{x - y}{\|x - y\|} = \frac{rx - ry}{\|rx - ry\|} \in S(S_X, f, \varepsilon),$$

which gives the inclusion $S(S_X, f_r, \varepsilon) \subset S(S_X, f, \varepsilon)$. The converse inclusion is proved analogously, or one just observes that $(f_r)_{1/r} = f$. \square

Proof of Lemma 2.4. Let y_1, y_2 be distinct elements in X such that $\frac{y_1 - y_2}{\|y_1 - y_2\|} \in \overline{\text{conv}}(A) \cap S(S_X, f, \varepsilon)$. By rescaling the functional, we can suppose that $\|y_1 - y_2\| = 1$: Indeed, let $r = \|y_1 - y_2\|$ and observe that

$$f_r\left(\frac{y_1}{\|y_1 - y_2\|}\right) - f_r\left(\frac{y_2}{\|y_1 - y_2\|}\right) = \frac{f(y_1) - f(y_2)}{\|y_1 - y_2\|}.$$

Taking into account Lemma 2.5, it is easy to observe that the functional f_r and the points $\frac{y_1}{\|y_1 - y_2\|}$, $\frac{y_2}{\|y_1 - y_2\|}$ satisfy the desired conditions.

Now, we have that

$$y_1 - y_2 \in \overline{\text{conv}}(A) \quad \text{and} \quad f(y_1) - f(y_2) > \|f\| - \varepsilon.$$

So we can find $x_1, \dots, x_n \in A$ and $\lambda_1, \dots, \lambda_n \in [0, 1]$ with $\sum_{k=1}^n \lambda_k = 1$ satisfying

$$f(y_1) - f(y_2) - \left\|y_1 - y_2 - \sum_{k=1}^n \lambda_k x_k\right\| \|f\| > \|f\| - \varepsilon$$

and, therefore,

$$f(y_1) - f(y_2) - \left|f(y_2) - f\left(y_1 - \sum_{k=1}^n \lambda_k x_k\right)\right| > \|f\| - \varepsilon.$$

We can write

$$\begin{aligned} & \frac{f(y_1) - f(y_1 - \lambda_1 x_1)}{\lambda_1} \lambda_1 + \frac{f(y_1 - \lambda_1 x_1) - f(y_1 - (\lambda_1 x_1 + \lambda_2 x_2))}{\lambda_2} \lambda_2 \\ & + \dots + \frac{f(y_1 - \sum_{k=1}^{n-1} \lambda_k x_k) - f(y_1 - \sum_{k=1}^n \lambda_k x_k)}{\lambda_n} \lambda_n \\ & = f(y_1) - f(y_2) + \left[f(y_2) - f\left(y_1 - \sum_{k=1}^n \lambda_k x_k\right)\right] \\ & > \|f\| - \varepsilon. \end{aligned}$$

Now, an evident convexity argument gives the existence of $\ell \in \{1, \dots, n\}$ such that

$$\frac{f\left(y_1 - \sum_{k=1}^{\ell-1} \lambda_k x_k\right) - f\left(y_1 - \sum_{k=1}^{\ell} \lambda_k x_k\right)}{\lambda_{\ell}} > \|f\| - \varepsilon,$$

understanding that in case $\ell = 1$ the element $\sum_{k=1}^{\ell-1} \lambda_k x_k$ is zero. Therefore, we get that

$$x_\ell = \frac{\left(y_1 - \sum_{k=1}^{\ell-1} \lambda_k x_k\right) - \left(y_1 - \sum_{k=1}^{\ell} \lambda_k x_k\right)}{\lambda_\ell} \in S(S_X, f, \varepsilon)$$

(recall that $\|x_\ell\| = 1$), which finishes the proof. \square

As a consequence of this result we can show that Lipschitz slices in a space with the Daugavet property present the same wild behaviour as linear ones.

Corollary 2.6. *Let X be a space with the Daugavet property, let $\varepsilon > 0$ and let S be a Lip-slice. Then, for every $x \in S_X$ there is $y \in S$ such that $\|x + y\| > 2 - \varepsilon$.*

Proof. Fix $x \in S_X$. Since X has the Daugavet property we have by [7, Lemma 2.2] that

$$\overline{\text{conv}}(\{y \in S_X : \|x + y\| > 2 - \varepsilon\}) = B_X;$$

so Lemma 2.4 gives that

$$\{y \in S_X : \|x + y\| > 2 - \varepsilon\} \cap S \neq \emptyset. \quad \square$$

A similar result for the alternative Daugavet property can be produced in the same way.

Corollary 2.7. *Let X be a space with the alternative Daugavet property, let $\varepsilon > 0$ and let S be a Lip-slice. Then, for every $x \in S_X$ there is $y \in S$ such that $\max_{\theta \in \mathbb{T}} \|x + \theta y\| > 2 - \varepsilon$.*

Proof. Fix $x \in S_X$. Since X has the alternative Daugavet property, we have by [12, Proposition 2.1] that

$$B_X = \overline{\text{conv}}(\mathbb{T}\{y \in S_X : \|x + y\| > 2 - \varepsilon\}) = \overline{\text{conv}}\left(\left\{y \in S_X : \max_{\theta \in \mathbb{T}} \|x + \theta y\| > 2 - \varepsilon\right\}\right),$$

so Lemma 2.4 gives the result. \square

3. THE DAUGAVET EQUATION FOR LIPSCHITZ OPERATORS

We devote this section to obtain some sufficient conditions for a Lipschitz operator to satisfy the Daugavet or the alternative Daugavet equation.

Recall from [1] that a bounded subset A of a Banach space X is an *SCD set* (SCD is an abbreviation for *slicely countably determined*) if there is a *determining sequence* $\{S_n : n \in \mathbb{N}\}$ of slices of A , i.e., a sequence $\{S_n : n \in \mathbb{N}\}$ such that $A \subset \overline{\text{conv}}(B)$ whenever $B \subset A$ intersects all the S_n 's. This property, which clearly implies separability, is possessed by many classes of separable bounded convex subsets, for example by dentable sets (in particular by Radon-Nikodým sets), by sets with the Asplund property, by strongly regular sets, by CPCP sets, by sets which do not contain ℓ_1 sequences [1] and by the unit ball of any space with a 1-unconditional basis [10]. Remark that in [1] the property SCD was defined only for convex sets, so for future applications of results from [1] we need the following simple lemma.

Lemma 3.1. *If $A \subset X$ and $\overline{\text{conv}}A$ is an SCD set, then A is also an SCD set.*

Proof. Let $\{S_n : n \in \mathbb{N}\}$ be a determining sequence of slices of $\overline{\text{conv}}A$. Then, the sets $S'_n := S_n \cap A$ are not empty and are slices of A . It remains to show that $\{S'_n : n \in \mathbb{N}\}$ is a determining sequence for A . This is evident: if $B \subset A$ intersects all the S'_n , then B intersects all the S_n , so $A \subset \overline{\text{conv}}A \subset \overline{\text{conv}}(B)$. \square

We need some more notation in the spirit of [1]. Given a Banach space X , we denote by $K(X^*)$ the intersection of S_{X^*} with the weak*-closure in X^* of $\text{ext}(B_{X^*})$. We consider $K(X^*)$ as a topological space equipped with the weak* topology $\sigma(X^*, X)$. For a Lip-slice S of S_X and $\varepsilon > 0$, we consider the set

$$\begin{aligned} D(S, \varepsilon) &:= \{x^* \in K(X^*) : \exists y \in S \text{ with } \text{Re } x^*(y) > 1 - \varepsilon\} \\ &= \{x^* \in K(X^*) : S \cap S(S_X, \text{Re } x^*, \varepsilon) \neq \emptyset\}. \end{aligned}$$

Lemma 3.2. *$K(X^*)$ is a Baire space and $D(S, \varepsilon)$ is an open subset of $K(X^*)$ for every $\varepsilon > 0$. If, moreover, X has the Daugavet property, then $D(S, \varepsilon)$ is dense in $K(X^*)$.*

Proof. Let $K'(X^*)$ be the weak*-closure in X^* of $\text{ext}(B_{X^*})$, which is weak*-compact, and observe that

$$K'(X^*) \setminus K(X^*) = \bigcup_{n \in \mathbb{N}} \left[\left(1 - \frac{1}{n}\right) B(X^*) \cap K'(X^*) \right]$$

is of the first category in $K'(X^*)$, so $K(X^*)$ is a Baire space.

Since each set $D_y := \{x^* \in K(X): \text{Re } x^*(y) > 1 - \varepsilon\}$ is relatively $\sigma(X^*, X)$ -open in $K(X^*)$, $D(S, \varepsilon) = \bigcup_{y \in S} D_y$, $D(S, \varepsilon)$ is evidently relatively $\sigma(X^*, X)$ -open in $K(X^*)$.

Finally, to show that $D(S, \varepsilon)$ is weak* dense in $K(X^*)$ for a Banach space with the Daugavet property it is sufficient to demonstrate that the weak* closure of $D(S, \varepsilon)$ contains every extreme point of B_{X^*} . Since weak*-slices form a base of (relative) neighborhoods of any extreme point of B_{X^*} (Choquet's lemma, see [4, Definition 25.3 and Proposition 25.13]), it is sufficient to prove that every weak*-slice

$$S(B_{X^*}, x, \delta) := \{x^* \in B_{X^*}: \text{Re } x^*(x) > 1 - \delta\}$$

with $\delta \in (0, \varepsilon)$ and $x \in S_X$ intersects $D(S, \varepsilon)$. To this end, let us use Corollary 2.6: Given x and δ as above, there is a $y \in S$ such that $\|x + y\| > 2 - \delta$. By Krein-Milman theorem, there is $y^* \in \text{ext}(B_{X^*})$ such that $\text{Re } y^*(x + y) > 2 - \delta$. Therefore, both $\text{Re } y^*(x) > 1 - \delta$ and $\text{Re } y^*(y) > 1 - \delta$, which implies that $y^* \in S(B_{X^*}, x, \delta) \cap D(S, \varepsilon)$. \square

An application of the Baire Theorem gives the following result.

Corollary 3.3. *Let X be a Banach space with the Daugavet property. Given any sequence of Lip-slices $\{S_n: n \in \mathbb{N}\}$ and any sequence $\{\delta_n: n \in \mathbb{N}\}$ of positive numbers, we have that $\bigcap_{n \in \mathbb{N}} D(S_n, \delta_n)$ is a dense G_δ -subset of $K(X^*)$.*

We are now ready to state the main result of the present section.

Theorem 3.4. *Let X be a Banach space with the Daugavet property. Then, every $T \in \text{Lip}(X)$ for which $\text{slope}(T)$ is an SCD-set satisfies (DE).*

Proof. It is sufficient to consider the case of $\|T\| = 1$. For $\varepsilon > 0$ fixed, let $u, v \in X$ with $u \neq v$ such that

$$\frac{\|T(u) - T(v)\|}{\|u - v\|} > 1 - \varepsilon.$$

Since $\text{slope}(T)$ is an SCD-set, there is a sequence $\{S_n\}$ of slices of $\text{slope}(T)$ which is determining. For every $n \in \mathbb{N}$, we write

$$\begin{aligned} S_n &= \{z \in \text{slope}(T): \text{Re } x_n^*(z) > 1 - \varepsilon_n\} \\ &= \left\{ \frac{T(x) - T(y)}{\|x - y\|}: x \neq y, \frac{\text{Re } x_n^*(T(x)) - \text{Re } x_n^*(T(y))}{\|x - y\|} > 1 - \varepsilon_n \right\}, \end{aligned}$$

where $\varepsilon_n > 0$ and $x_n^* \in S_{X^*}$. Next, for each $n \in \mathbb{N}$ we consider the subset of S_X given by

$$\begin{aligned} \widetilde{S}_n &= \left\{ \frac{x - y}{\|x - y\|}: x \neq y, \frac{T(x) - T(y)}{\|x - y\|} \in S_n \right\} \\ &= \left\{ \frac{x - y}{\|x - y\|}: x \neq y, \frac{\text{Re } x_n^*(T(x)) - \text{Re } x_n^*(T(y))}{\|x - y\|} > 1 - \varepsilon_n \right\} \end{aligned}$$

and observe that \widetilde{S}_n is not empty (as S_n is a non-empty subset of $\text{slope}(T)$) and so, as $\text{Re } x_n^* \circ T \in \text{Lip}(X, \mathbb{R})$, \widetilde{S}_n is a Lip-slice of S_X . Therefore, we can apply Corollary 3.3 for the sequence $\{\widetilde{S}_n: n \in \mathbb{N}\}$ to obtain that the set

$$A = \bigcap_{n \in \mathbb{N}} D(\widetilde{S}_n, \varepsilon)$$

is dense in $K(X^*)$. Thus, there is $x^* \in A$ such that

$$\operatorname{Re} x^* \left(\frac{T(u) - T(v)}{\|u - v\|} \right) > 1 - \varepsilon. \quad (3)$$

Besides, since $x^* \in A$, for every $n \in \mathbb{N}$ we have that $S(S_X, \operatorname{Re} x^*, \varepsilon) \cap \tilde{S}_n \neq \emptyset$. Hence, for every $n \in \mathbb{N}$ we can find distinct $x_n, y_n \in X$ such that

$$\frac{x_n - y_n}{\|x_n - y_n\|} \in S(S_X, \operatorname{Re} x^*, \varepsilon) \quad \text{and} \quad \frac{T(x_n) - T(y_n)}{\|x_n - y_n\|} \in S_n.$$

Therefore, using that the sequence $\{S_n\}$ is determining for $\operatorname{slope}(T)$, we get that

$$\operatorname{slope}(T) \subset \overline{\operatorname{conv}} \left\{ \frac{T(x_n) - T(y_n)}{\|x_n - y_n\|} : n \in \mathbb{N} \right\}.$$

This, together with (3), allows us to find $\lambda_1, \dots, \lambda_N \geq 0$ with $\sum_{k=1}^N \lambda_k = 1$ such that

$$\operatorname{Re} x^* \left(\sum_{k=1}^N \lambda_k \frac{T(x_k) - T(y_k)}{\|x_k - y_k\|} \right) > 1 - \varepsilon.$$

So, by convexity, there is $\ell \in \{1, \dots, N\}$ satisfying

$$\operatorname{Re} x^* \left(\frac{T(x_\ell) - T(y_\ell)}{\|x_\ell - y_\ell\|} \right) > 1 - \varepsilon.$$

Recalling that $\frac{x_\ell - y_\ell}{\|x_\ell - y_\ell\|} \in S(S_X, \operatorname{Re} x^*, \varepsilon)$, we can write

$$\|\operatorname{Id} + T\| \geq \left\| \frac{x_\ell - y_\ell}{\|x_\ell - y_\ell\|} + \frac{T(x_\ell) - T(y_\ell)}{\|x_\ell - y_\ell\|} \right\| \geq \operatorname{Re} x^* \left(\frac{x_\ell - y_\ell}{\|x_\ell - y_\ell\|} + \frac{T(x_\ell) - T(y_\ell)}{\|x_\ell - y_\ell\|} \right) > 2 - 2\varepsilon.$$

Letting $\varepsilon \rightarrow 0$, we get $\|\operatorname{Id} + T\| \geq 2$, which finishes the proof since the converse inequality always holds. \square

The condition that $\operatorname{slope}(T)$ is an SCD-set of Theorem 3.4 means, in particular, that T has separable image. In order to get rid of this separability restriction, one can use the following lemma.

Lemma 3.5. *Let X be a Banach space with the Daugavet property and let $T \in \operatorname{Lip}(X)$. Then there is a separable subspace $E \subset X$ having the Daugavet property, with $T(E) \subset E$ and $\|T|_E\| = \|T\|$.*

Proof. Fix a sequence $(x_k) \subset X$ such that

$$\lim_{k \rightarrow \infty} \frac{\|T(x_{2k}) - T(x_{2k-1})\|}{\|x_{2k} - x_{2k-1}\|} = \|T\| \quad (4)$$

and define recursively two sequences of separable subspaces $X_n \subset X$ and $E_n \subset X$,

$$X_1 \subset E_1 \subset X_2 \subset E_2 \dots,$$

in the following way. Put $X_1 = \overline{\operatorname{lin}}\{x_k\}_{k \in \mathbb{N}}$. When a separable subspace X_n is defined, we select a separable subspace $E_n \supset X_n$ having the Daugavet property (the possibility of such a choice is guaranteed by [11, Theorem 4.5]) and put $X_{n+1} = \overline{\operatorname{lin}}(E_n \cup T(E_n))$. Under this construction $E = \overline{\bigcup_{n \in \mathbb{N}} E_n}$ is the subspace we need (the Daugavet property of E easily follows from the Daugavet property of the E_n and from Lemma 2.1). \square

Corollary 3.6. *Let X be a Banach space with the Daugavet property, $T \in \operatorname{Lip}(X)$ and suppose that $\overline{\operatorname{conv}}(\operatorname{slope}(T))$ has one of the following properties: Radon-Nikodým property, Asplund property, CPCP or absence of ℓ_1 -sequences. Then T satisfies (DE).*

Proof. Let $E \subset X$ be the subspace from Lemma 3.5. Denote by $T_E \in \text{Lip}(E)$ the restriction of T to E . Then $\overline{\text{conv}}(\text{slope}(T_E)) \subset \overline{\text{conv}}(\text{slope}(T))$, so $\overline{\text{conv}}(\text{slope}(T_E))$ also has one of the properties listed above and hence, as it is also separable, $\overline{\text{conv}}(\text{slope}(T_E))$ is an SCD-set; see [1]. Thanks to Lemma 3.1 this means that $\text{slope}(T_E)$ is an SCD-set, so by Theorem 3.4 T_E satisfies the Daugavet equation. Consequently,

$$\|\text{Id} + T\| \geq \|\text{Id}_E + T_E\| = 1 + \|T_E\| = 1 + \|T\|,$$

as claimed. \square

Similar results like Theorem 3.4 and Corollary 3.6 hold true for the alternative Daugavet property as well.

Theorem 3.7. *Let X be a Banach space with the alternative Daugavet property. Then, every $T \in \text{Lip}(X)$ for which $\text{slope}(T)$ is an SCD-set satisfies (aDE).*

Corollary 3.8. *Let X be a Banach space with the alternative Daugavet property, $T \in \text{Lip}(X)$ and suppose that $\overline{\text{conv}}(\text{slope}(T))$ has one of the following properties: Radon-Nikodým property, Asplund property, CPCP or absence of ℓ_1 -sequences. Then T satisfies (aDE).*

To get these two results, one only needs to modify the definition of $D(S, \varepsilon)$ and to generalize Lemma 3.2, Corollary 3.3 and Lemma 3.5. We state here such modified results but we omit their proofs which are just adaptations of the corresponding ones for the Daugavet property, as it was done in [1, §4 & §5] in the linear case. We need some notation. For a Lip-slice S and $\varepsilon > 0$, we consider the set

$$\begin{aligned} \tilde{D}(S, \varepsilon) &:= \{x^* \in K(X^*) : \exists y \in S \text{ with } |x^*(y)| > 1 - \varepsilon\} \\ &= \{x^* \in K(X^*) : S \cap \text{TS}(S_X, \text{Re } x^*, \varepsilon) \neq \emptyset\}. \end{aligned}$$

Lemma 3.9. *$\tilde{D}(S, \varepsilon)$ is an open subset of $K(X^*)$. If, moreover, X has the alternative Daugavet property, then $\tilde{D}(S, \varepsilon)$ is dense in $K(X^*)$.*

Corollary 3.10. *Let X be a Banach space with the alternative Daugavet property. Given any sequence of Lip-slices $\{S_n : n \in \mathbb{N}\}$ and any sequence $\{\delta_n : n \in \mathbb{N}\}$ of positive numbers, we have that $\bigcap_{n \in \mathbb{N}} \tilde{D}(S_n, \delta_n)$ is a dense G_δ -subset of $K(X^*)$.*

Lemma 3.11. *Let X be a Banach space with the alternative Daugavet property and let $T \in \text{Lip}(X)$. Then there is a separable subspace $E \subset X$ having the alternative Daugavet property, with $T(E) \subset E$ and $\|T|_E\| = \|T\|$.*

Let us remark that as a consequence of Theorem 3.7 and Corollary 3.8, we get the following results, which extend [16, Theorem 3.4] in the case of $n(X) = 1$. They are also consequences of the results of the next section and, in the real case, of [16, Theorem 2.6] and [1, Theorem 4.4]. Recall that a Banach space X is *SCD* [1] if every bounded convex subset of X is SCD (by Lemma 3.1 this is equivalent to the fact that every bounded subset of X is SCD). Examples of SCD spaces are separable Asplund spaces, separable spaces with the Radon-Nikodým property, those separable spaces not containing ℓ_1 , and separable spaces with the convex point of continuity property.

Corollary 3.12. *Let X be an SCD Banach space. If $n(X) = 1$, then $n_L(X) = 1$.*

As a consequence of Corollary 3.8, we have the following result.

Corollary 3.13. *Let X be a Banach space with $n(X) = 1$. If X has the Radon-Nikodým property, or X is an Asplund space, or X does not contain ℓ_1 , or X has the convex point of continuity property, then $n_L(X) = 1$.*

4. COMPLEX LUSH SPACES AND LIPSCHITZ NUMERICAL INDEX

Since it is not easy to deal with Banach spaces with numerical index 1, several sufficient geometrical conditions have been considered in the literature (see [8]), the weakest one being the so-called lushness. A Banach space X is said to be *lush* [3] if for every $x, y \in S_X$ and every $\varepsilon > 0$, there is a slice $S = S(S_X, \operatorname{Re} y^*, \varepsilon)$ with $y^* \in S_{X^*}$ such that $y \in S$ and $\operatorname{dist}(x, \operatorname{conv}(\mathbb{T}S)) < \varepsilon$ (observe that the original definition of lushness used slices of the unit ball, but this reformulation is equivalent.) Lush spaces have numerical index 1 [3, Proposition 2.2], but the converse result is not true [6], even though most of the known examples of Banach spaces with numerical index 1 are actually lush [2]. We refer to the cited papers [2, 3, 6] for background.

It is proved in [16, Theorem 2.6] that real lush spaces have Lipschitz numerical index 1. Our aim is to show that the same happens for complex spaces.

Theorem 4.1. *Let X be a (complex) lush space. Then, $n_L(X) = 1$.*

We need a reformulation of lushness which follows from the results of [9].

Lemma 4.2. *Let X be a lush space. Then, for every $x, y \in S_X$ and every $\varepsilon > 0$, there is a slice $S = S(S_X, \operatorname{Re} y^*, \varepsilon)$ with $y^* \in S_{X^*}$ such that $y \in S$ and $x \in \overline{\operatorname{conv}}(\mathbb{T}S)$.*

Proof. By Lemma 4.2 of [9], given $y \in S_X$ and $\varepsilon > 0$, there is a dense subset K_y of $K(X^*)$ (with the notation of section 3) such that $y \in \overline{\operatorname{conv}}(\mathbb{T}S(S_X, \operatorname{Re} y^*, \varepsilon))$ for every $y^* \in K_y$ (observe that in [9, Lemma 4.2], K_y is a subset of what is called $K'(X^*)$ in the proof of Lemma 3.2, but since $K(X^*)$ is also a Baire space, the same argument gives the result that we are using). Now, as the set of extreme points of B_{X^*} is contained in the closure of K_y , there is $y^* \in K_y$ such that $\operatorname{Re} y^*(x) > 1 - \varepsilon$, that is, $x \in S(S_X, \operatorname{Re} y^*, \varepsilon)$. \square

Proof of 4.1. Let $T \in \operatorname{Lip}(X)$ with $\|T\| = 1$. By [16, Corollary 2.3] it suffices to show that $\max_{\theta \in \mathbb{T}} \|\operatorname{Id} + \theta T\| = 2$ (actually, it is equivalent). For a fixed $\varepsilon > 0$, there are $x_1, x_2 \in X$ such that

$$\frac{\|T(x_1) - T(x_2)\|}{\|x_1 - x_2\|} > 1 - \varepsilon.$$

We denote

$$y = \frac{T(x_1) - T(x_2)}{\|x_1 - x_2\|}, \quad y_0 = \frac{y}{\|y\|}, \quad \text{and} \quad x_0 = \frac{x_1 - x_2}{\|x_1 - x_2\|}.$$

Using Lemma 4.2, we may find a slice $S = S(S_X, \operatorname{Re} y^*, \varepsilon)$ such that $y_0 \in S$ and $x_0 \in \overline{\operatorname{conv}}(\mathbb{T}S)$. Next, we write

$$\tilde{S} = \left\{ \frac{y_1 - y_2}{\|y_1 - y_2\|} : y_1 \neq y_2, \frac{\operatorname{Re} y^* \circ T(y_1) - \operatorname{Re} y^* \circ T(y_2)}{\|y_1 - y_2\|} > 1 - 2\varepsilon \right\}$$

and observe that $x_0 \in \tilde{S}$ since we have that

$$\frac{\operatorname{Re} y^*(T(x_1)) - \operatorname{Re} y^*(T(x_2))}{\|x_1 - x_2\|} = \operatorname{Re} y^*(y) > \operatorname{Re} y^*(y_0) - \varepsilon > 1 - 2\varepsilon.$$

In particular, \tilde{S} is not empty and so it is a Lip-slice. Further, $x_0 \in \overline{\operatorname{conv}}(\mathbb{T}S) \cap \tilde{S}$, so Lemma 2.4 gives us that $\mathbb{T}S$ intersects \tilde{S} as well. Hence, there are $\theta_0 \in \mathbb{T}$ and $z \in S$ satisfying $\theta_0 z \in \tilde{S}$. That is, there exist $z_1 \neq z_2$ in X such that

$$\theta_0 z = \frac{z_1 - z_2}{\|z_1 - z_2\|} \quad \text{and} \quad \frac{\operatorname{Re} y^* T(z_1) - \operatorname{Re} y^* T(z_2)}{\|z_1 - z_2\|} > 1 - 2\varepsilon.$$

Finally, we have

$$\begin{aligned}
\max_{\theta \in \mathbb{T}} \|\text{Id} + \theta T\| &\geq \|\theta_0^{-1} \text{Id} + T\| \\
&\geq \left\| z + \frac{Tz_1 - Tz_2}{\|z_1 - z_2\|} \right\| \\
&\geq \text{Re } y^* \left(z + \frac{Tz_1 - Tz_2}{\|z_1 - z_2\|} \right) \\
&= \text{Re } y^*(z) + \text{Re } y^* \left(\frac{Tz_1 - Tz_2}{\|z_1 - z_2\|} \right) \\
&> (1 - \varepsilon) + (1 - 2\varepsilon) = 2 - 3\varepsilon,
\end{aligned}$$

which finishes the proof by letting $\varepsilon \rightarrow 0$. □

As we have already commented in the previous section, Corollaries 3.12 and 3.13 also follow from this result.

It is asked in [16] whether $n(X) = n_L(X)$ for every Banach space X . As far as we know, even the following particular case of the above question also remains open.

Question 4.3. Let X be a Banach space with $n(X) = 1$. Is it true that $n_L(X) = 1$?

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